2D Transformation Matrices!

A 2×2 matrix $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be used to transform any point $P=(x,y)$ in the 2D Cartesian plane as follows:

First P is written as a column matrix $\begin{bmatrix} x \\ y \end{bmatrix}$. This column matrix is then pre-multiplied by A:

$$A\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$

The resulting matrix $\begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$ gives the coordinates of the image $P'=(ax+by,cx+dy)$.

Thus the matrix A can be used to make the transformation

$$(x,y)\to(ax+by,cx+dy).$$

Go to

https://www.desmos.com/calculator/erl8wdtrvu

Click the spanner button at the top right and choose the large ‘A’ to increase the display size.

Here you can investigate the transformations of a square that result from multiplying the coordinates of its points by a matrix $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Initially A should be the identity matrix (i.e. $a=1, b=0, c=0$ and $d=1$).

You will now investigate in turn the effects of changing the values of a, b, c and d. (Note that for any of these tasks you can choose to show a ‘Reference square’ which will be unchanged by the transformations.)

1. Use the slider to change just the values of a (keeping $b=0$, $c=0$ and $d=1$) and describe as precisely as possible the resulting transformations of the square.

The transformations above result from the following equation:

$$\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} ax \\ y \end{bmatrix}$$

giving transformations defined by

$$(x, y) \rightarrow (ax, y)$$

2. With reference to the above transformation definition, explain your answer to Question 1.

Matrix form	Effect on the square	Transformation	Explanation
$\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$	The square is dilated horizontally by a factor of $\|a \vee$¿. If $a<0$ then the square is reflected through the y-axis.	$(x, y) \rightarrow (ax, y)$	x-coordinates are multiplied by a, resulting in a horizontal dilation by a, but y-coordinates are unchanged.
$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$			
$\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$			

$\begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix}$			

Now, by adjusting values of a, b, c and c, try to **rotate** the square by $90°$ anti-clockwise.

4. Complete the following table.

Rotation angle (anticlockwise)	**Matrix**	**Transformation definition**
$90°$		
$180°$		
$270°$		
$45°$		
$135°$		
$30°$		
$60°$		

5. What relationship exists between the rotation angle θ and the values of a, b, c and d?

6. Write down, in terms of θ, the general form of a matrix which will rotate the square anticlockwise by the angle θ.

Now, by adjusting values of a, b, c and d, try to **reflect** the square through the y-axis.

7. Investigate reflections through lines passing through the origin at various angles to the x-axis, and complete the following table. To help with this you can click to show the given line of reflection and adjust its angle θ (given in Desmos as T) from the x-axis. You may also find it helpful to show the Reference square.

Angle of line of reflection from x-axis	**Matrix**	**Transformation definition**
$90°$		
$45°$		
$0°$		
$30°$		
$60°$		
$120°$		
$135°$		

8. What relationship exists between the angle θ of the line of reflection and the values of a, b, c and d?

9. Write down, in terms of θ, the general form of a matrix which will reflect the square through a line passing through the origin at an angle θ from the x-axis.